MATHEMATICS

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MATHEMATICS
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COMPLEX NUMBER

& Their Properties

CONTENTS

THINGS TO REMEMBER

✷ Complex Numbers

The number of the form $x + iy$ are known as complex numbers. Here x and y are real numbers and $i = \sqrt{-1}$ is iotc.

The complex number is usally denoted by z and its set is denoted by C.

$\therefore$ $$C = \{x + iy : x, y \in R, i = \sqrt{-1}\}.$$

eg, $7 + 2i$, $0 + i$, $1 + 0i$, etc are complex numbers.

Integral Power of Iota : $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.

In general $i^{4n} = 1$, $i^{4n+1} = -i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, for any integer n

eg, $$i^{1998} = i^{4 \times 499 + 2} = -1.$$

Real and Imaginary Parts of Complex Number

Let $z = x + iy$ is a complex number, then x is called the real part of z and is denoted by Re(z) and y is called the imaginary part of z and is denoted by Im(z).

eg, If $z = 7 + 4i$, then Re(z) = 7 and Im(z) = 4.

A complex number z is said to be purely real if Im(z) = 0 and is said to be purely imaginary if Re(z) = 0. The complex number $0 = 0 + i0$ is both purely real and purely imaginary.

Every real number 'a' can be written as $a + i0$. Therefore, every real number is considered as a complex number whose imaginary part is zero.

✷ Equality of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then these two numbers re equal, if

$$x_1 = x_2 \text{ and } y_1 = y_2$$

ig, $$\text{Re}(z_1) = \text{Re}(z_2)$$

and $$\text{Im}(z_1) = \text{Im}(z_2)$$

eg, If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal, then $2 - iy = x + 3i$.

$\Rightarrow$ $$x = 2 \text{ and } y = -3$$

✷ Algebraic Operations on Complex Numbers

1. Addition of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex number then

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

$\Rightarrow$ $$\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$$

and $$\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$$

Properties of Addition of Complex Number

(i) $z_1 + z_2 = z_2 + z_1$ (Commutative law)

(ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (Associative law)

(iii) $z + 0 = 0 + z$ (where $0 = 0 + i0$)

2. **Subtraction of Complex Number**

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex number then

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

$\Rightarrow$ $\mathrm{Re}(z_1 - z_2) = \mathrm{Re}(z_1) - \mathrm{Re}(z_2)$

and $\mathrm{Im}(z_1 - z_2) = \mathrm{Im}(z_1) - \mathrm{Im}(z_2)$

3. **Multiplication of Complex Number**

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex number then

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 x_2 + y_1 y_2)$$

$\Rightarrow$ $z_1 z_2 = [\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\,\mathrm{Im}(z_2)] + i[\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) + \mathrm{Im}(z_1)\,\mathrm{Im}(z_2)]$

Properties of Addition of Complex Number

(i) $z_1 z_2 = z_2 z_1$ (Commutative law)

(ii) $z_1 (z_2 z_3) = (z_1 z_2) z_3$ (Associative law)

(iii) If $z_1 z_2 = 1 = z_2 z_1$, then z_1 and z_2 are multiplicative inverse of each other.

(iv) (a) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ (Left distribution law)

(b) $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$ (Right distribution law)

4. **Division of Complex Number**

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex number then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

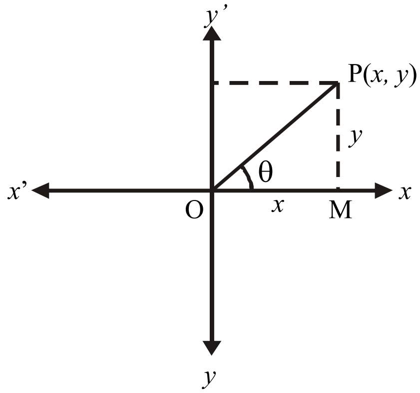
$$= \frac{1}{x_2^2 + y_2^2}\left[(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)\right]$$

✷ **<u>Representation of a Complex Numbers</u>**

Comlex number can be represented as follows

1. **Geometrical Representation of a Complex Number**

The complex number may be represented graphically by the oint P whose rectangular coordinated are (x, y). Thus each point in the plane is associated with a complex number. In th figure P defines $z = x + iy$. it is customary to choose x-axis as real axis and y-axis as imagnary axis. Such a plane is called Argand plane or Argand diagram of complex plane or gaussian plane.



Distance of P from origin is OP = $\sqrt{x^2+y^2}$. It is called the modulus of z and angle of OP with positive direction of x-axis is called argument of z.

$\therefore$ $$\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

2. Trigonometrical or Polar Representation of a Complex Number

Let $z = x + iy$ is a complex number which is denoted by a point P(x, y) in a complex plane, then

OP = | z |

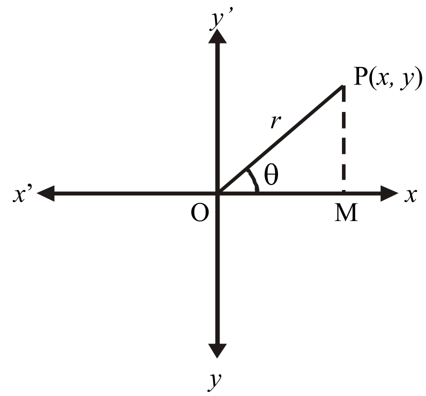
and POX = t = arg (z)

In POM,

$$\cos\theta = \frac{OM}{OP} = \frac{x}{|z|}$$

$\Rightarrow$ x = | z | cos θ

and $$\sin\theta = \frac{PM}{OP} = \frac{y}{|z|}$$



$\Rightarrow$ x = | z | cos θ

$\therefore$ $z = x + iy$

$\Rightarrow$ z = | z | cosθ + i | z | sinθ

$\Rightarrow$ z = | z | (cosθ + i sinθ)

$\Rightarrow$ z = r (cosθ + i sinθ)

Where r = | z | and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

This form of z is known as polar form.

In general, polar form is

z = r [cos(2$n\pi$ + θ) + i sin(2$n\pi$ + θ)]

Where, r = | z |, θ = arg(z) and n ∈ N.

3. Eulerian Form of a Complex Number

We have, $e^{i\theta}$ = cosθ + i sinθ and e^{-ie} = cosθ – i sinθ

These two are called Euler's notations.

Let z be any complex number number such that | z | = r and arg(z) = t. Then z = $z = x + iy$ = r(cosθ + i sinθ) can be represented in exponential or Eulerian form as.

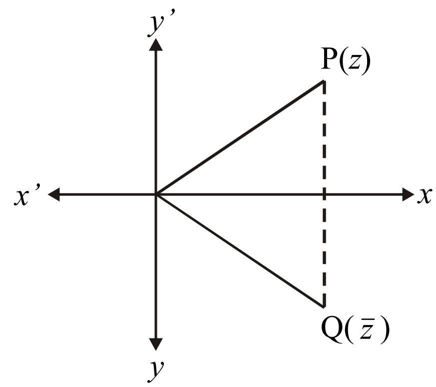
$$z = r e^{i\theta} = r(\cos\theta + i\sin\theta)$$

✷ Conjugate of a Complex Number

Let z is a complex Number, then conjugate of z is denoted by $\bar{z}$ of z' and is equal to $x - i$y.

Thus, $\bar{z} = x - iy$

Geometrically, the conjugate of z is the reflection of point image of z in the real axis.



eg, If z = 3 + 4i, then $\bar{z} = 3 - 4i$

Properties of Conjuate of Complex Numbers

If z, z_1, z_2 are complex numbers, then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}\,(z)$

(iii) $z - \bar{z} = 2\,\text{Im}\,(z)$

(iv) $z = \bar{z} \Rightarrow$ z is purely real.

(v) $z = -\bar{z} \Rightarrow$ z is purely imaginary.

(vi) $z\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2$

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

(x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2},\ z_2 \neq 0$

(xi) $\bar{z}^n = (\bar{z})^n$

✷ Modulus of a Complex Number

Let $z = x + iy$ is a complex Number, then modulus of a complex number z is denoted by | z |.

$$\therefore \quad |z| = \sqrt{x^2 + y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$$

eg, If z = 4 + 3*i* is a complex number, then

$$|z| = \sqrt{4^2 + 3^2}$$
$$= \sqrt{16+9} = \sqrt{25} = 5$$

Properties of Modulus of Complex Numbers

(i) $|z| \geq 0 - |z| = 0$, iff $z = 0$ and $|z| > 0$, iff $z \neq 0$

(ii) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iii) $|z| = |\bar{z}| = |-z| = |\bar{z}|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

In general, $|z_1 z_2 z_3 .....z_n| = |z_1||z_2||z_3| ...... |z_n|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $(z_2 \neq 0)$

(vii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general, $|z_1 \pm z_2 \pm z_3 \pm ..... \pm z_n| \leq |z_1| + |z_2| + |z_3| + ... + |z_n|$

(viii) $|z_1 \pm z_2| \geq ||z_1| + |z_2||$

(ix) $|z^n| = |z|^n$

(x) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq ||z_1| + |z_2||$

Thus, $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and | z1 | – | z2 | is the least possible value of $|z_1 + z_2|$.

(xi) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2)$

$= |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$

$= |z_1|^2 + |z_2|^2 \pm \text{Re}(z_1\bar{z}_2)$

$= |z_1|^2 + |z_2|^2 \pm 2|z_1||\bar{z}_2| \cos(\theta_1 - \theta_2)$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

Where, $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

(xiii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(xiv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xv) $|az_1 - bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

✷ Argument of a Complex Number

Let $z = x + iy$ is a complex Number, then argument of complex number is dinoted by arg(z) or amp(z).

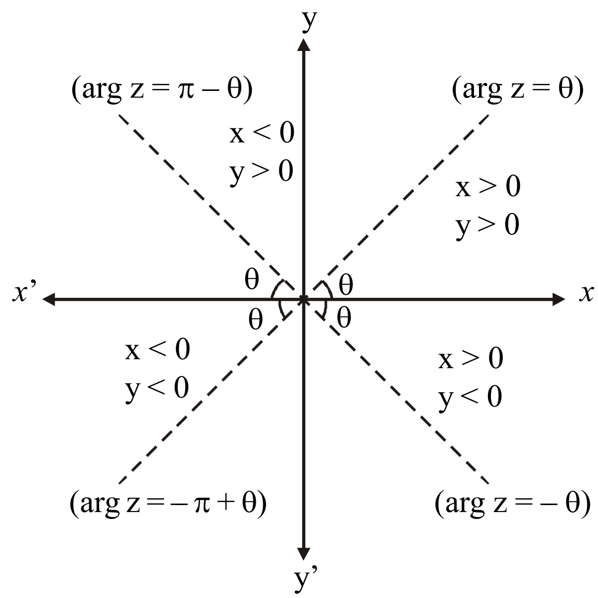
$$\arg(z) = \tan^{-1}\left|\frac{y}{x}\right|.$$

eg, If z = 4 + 3i is a complex number, then arg(z) = $\tan^{-1}\left(\frac{3}{4}\right)$.

Principal Value of Argument

The value of t of the argument which satisfies the inequality $-\pi < \theta < \pi$ is called the principal value of the argument.

Principal values of the argument are θ, $\pi - \theta$, $-\pi + \theta$, $-\theta$ according as the complex number lies on the I[st], II[nd], III[rd] and IV[th] qudrant.



Properties of Argument of Complex Numbers

If z_1, z_2, z_3 are three complex numbers, then

(i) $\arg(z_1, z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$ ($k = 0$ or 1 or -1)

In general, $\arg(z_1z_2z_3.....z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) ...... \arg(z_n) + 2k\pi$ ($k = 0$ or 1 or -1)

(ii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$ ($k = 0$ or 1 or -1)

(iii) $\arg\left(\dfrac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi$ ($k = 0$ or 1 or -1)

(iv) $\arg(z^n) = n\arg(z) + 2k\pi$ ($k = 0$ or 1 or -1)

(v) If $\arg\left(\dfrac{z_2}{z_1}\right) = \theta$, then $\arg\left(\dfrac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$

(vi) $\arg(\bar{z}) = -\arg(z)$

(vii) If $\arg(z) = 0 \Rightarrow z$ is real.

(viii) $\arg(z_1\bar{z}_2) = \arg(z_1) - \arg(z_2)$

(ix) $|z_1 + z_2| = |z_1 - z_2|$

$$\Rightarrow \quad \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$$

(x) $|z_1 + z_2| = |z_1| + |z_2|$

$$\Rightarrow \quad \arg(z_1) = \arg(z_2)$$

(xi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow \quad \dfrac{z_1}{z_2}$ is purely imaginary.

(xii) If $|z_1| \le 1, |z_2| \le 1$, then

(a) $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$

(b) $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - [\arg(z_1) - \arg(z_2)]^2$

✷ Square Root of a Complex Number

Let $a + ib$ is a complex Number such that $\sqrt{a+ib} = x + iy$, where x and y are real numbers.

Now, $\sqrt{a+ib} = x + iy \quad \Rightarrow \quad (x + iy)^2 = a + ib$

$\Rightarrow \quad (x^2 - y^2) + 2ixy = a + ib$

$\Rightarrow \quad x^2 - y^2 = a \quad$ ....(i)

and $\quad 2xy = b \quad$ ....(ii)

Now, $\quad (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$\Rightarrow \quad (x^2 + y^2)^2 = a^2 + b^2$

$\Rightarrow \quad (x^2 + y^2) = \sqrt{a^2 + b^2} \quad$ ....(iii)

$[\because x^2 + y^2 > 0]$

On solving Eqs. (i) and (iii), we get

$$x^2 = \left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2} + a\right]$$

and $$y^2 = \left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2} - a\right]$$

$\Rightarrow$ $$x = \pm\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2} + a\right]}$$

and $$y = \pm\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2} - a\right]}$$

If b is positive, then the sign of x and y from Eq. (ii) will be same ie,

$$\sqrt{a+ib} = \pm\left[\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2} + a\right]} + i\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2} - a\right]}\right]$$

If b is negative, then the sign of x and y will be opposite.

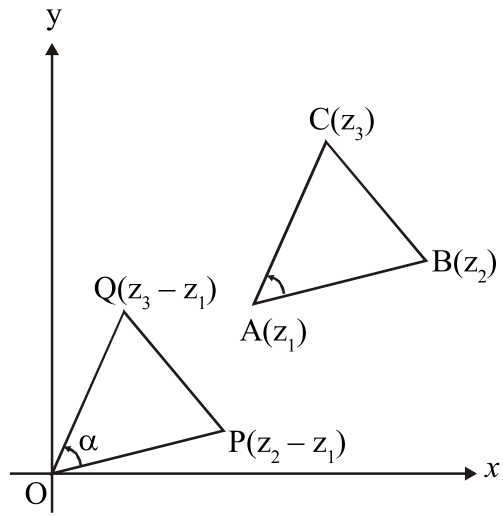
ie, $$\sqrt{a+ib} = \pm\left[\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2} + a\right]} - i\sqrt{\left(\frac{1}{2}\right)\left[\sqrt{a^2+b^2} - a\right]}\right]$$

✷ Concept of Rotation

Let z_1, z_2, z_3 be the vertices of a triangle ABC discribed in anti clockwise sense. Draw OP and OQ parallel and equal to AB and AC respectively. Then point P is $z_2 - z_1$ and Q is $z_3 - z_1$. If OP is rotated through angle α in anti-clockwise sense it coincides with $\overrightarrow{OQ}$.

[$\because$ OPQ and ABC are congruent. $\therefore \dfrac{OQ}{OP} = \dfrac{CA}{BA}$]

or $$\text{amp}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \alpha$$



✷ DeMoivre's Theorem

A simple formula for calculating powers of complex number known as De Moivre's Theorem. If n is a rational number, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Application of DeMoivre's Theorem

1. If $z = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots\ldots\ldots\ldots\ldots\ldots (\cos\theta_n + i\sin\theta_n)$

   then, $z = \cos(\theta_1 + \theta_2 + \ldots\ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots\ldots\ldots + \theta_n)$

2. If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

   $$(z)^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right]$$

   where, $k = 0, 1, 2, 3, \ldots\ldots\ldots\ldots, (n-1)$

3. $(\cos\theta - i\sin\theta) = \cos n\theta - i\sin n\theta$

4. $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

5. $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

6. $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)\right]^n$

$$= \left[\cos\left(\frac{n\pi}{2} - n\theta\right)\right] + i\left[\sin\left(\frac{n\pi}{2} - n\theta\right)\right]$$

7. $(\cos\theta + i \sin \phi)^n \neq \cos n\theta \pm i \sin n\phi$

✷ Cube Roots of Unity

Let $x = \sqrt[3]{1} \Rightarrow x^3 - 1 = 0$

$\Rightarrow (x - 1)(x^2 + x + 1) = 0$

Therefore, $x = 1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$

If second root be represented by w, then third root will be ω^2.

$\therefore$ Cube roots of unity are 1, ω, ω^2, 1 is a real root of unity and other two ie, ω and ω^2 are conjugate complex of each other.

Properties of Cube Roots of Unity

(i) $\omega^3 = 1$ or $\omega^{3r} = 1$

(ii) $\omega^{3r+1} = \omega$, $\omega^{3r+2} = \omega^2$

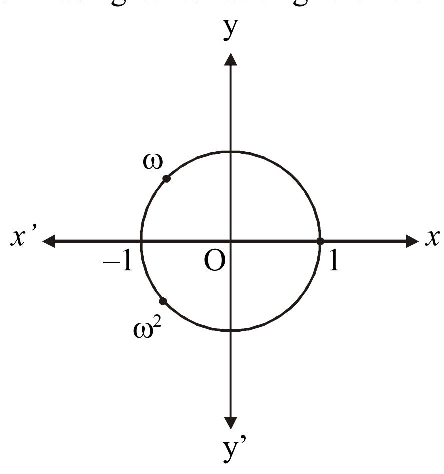
(iii) $1 + \omega^r + \omega^{2r} = 0$, if 'r' in not a multiple of '3'

$= 3$, if 'r' is multiple of '3'.

(iv) Each complex cube root of unity is square of other and also reciprocal of each other.

(v) If a is any positive number, then $a^{1/3}$ has roots $a^{1/3}$ (1), $a^{1/3}$ (ω), $a^{1/3}$ (ω^2) and if a is any negative number, then $a^{1/3}$ has roots $-|a|^{1/3}$, $|a|^{1/3}\omega$, $|a|^{1/3}\omega^2$.

(vi) The Cube roots of unity when represented on complex plane lie on vertices of an equilateral tri angle inscribed in a unit circle having center at origin. One vertex being on positive real axis.



Important Relations

(i) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(ii) $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$

(iii) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(iv) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

✷ *n*th Roots of Unity

Let $z = 1^{1/n}$, then

$z = (\cos 0^o + i \sin 0^o)^{1/n}$

$\Rightarrow \quad z = (\cos 2r\pi + i\sin 2r\pi)^{1/n}, r \in I$

$\Rightarrow \quad z = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$, r = 0, 1, 2,.............., (n – 1)

[Using DeMoivre's theorem]

$\Rightarrow \quad z = \left\{e^{\frac{i2\pi}{n}}\right\}^r$, r = 0, 1, 2,.............., (n – 1)

$\Rightarrow \quad z = \alpha^r, \alpha = e^{\frac{i2\pi}{n}}$, r = 0, 1, 2,.............., (n – 1)

Thus, *n*th roots of unity are 1, α, α^2,.................α^{n-1}, where $\alpha = e^{\frac{i2\pi}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

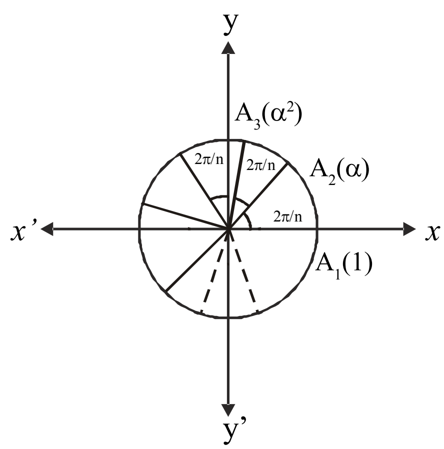
Properties of *n*th Roots of Unity.

(i) *n*th roots of unity form form a GP with common ratio.

(ii) Sum of nth roots of unity is always zero.

(iii) Product of nth roots of unity is $(-1)^{n-1}$.

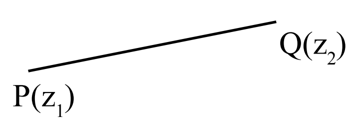
(iv) *n*th roots of unity are the vertices of the regular polygon of n sides inscribed in a circle of radius unity centred at origion. One vertex being on the positive real axis.



✷ <u>**Use of Complex Numbers in Coordinate Geometry**</u>

1. Distance between Two Points

Distance between two points $P(z_1)$ and $Q(z_2)$ is

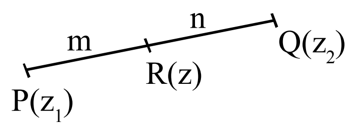


$PQ = |z_2 - z_1|$

2. Section Formula

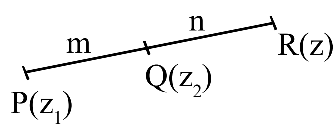
Let R(z) divides a join of P(z1) and Q(z2) in the ratio m : n

(i) If R(z) divides the line segment PQ internally, then



$$z = \frac{mz_2 + nz_1}{m+n}$$

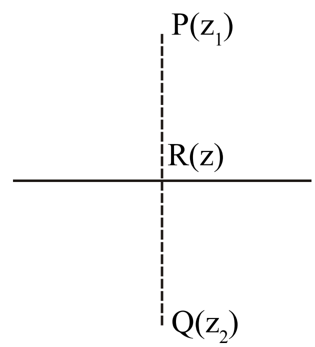
(ii) If R(z) divides the line segment PQ externally, then



$$z = \frac{mz_2 + nz_1}{m+n}$$

3. Equation of Perpendicular Bisector

If $P(z_1)$ and $Q(z_2)$ are two fixed points ard R(z) is an equidistant point from P and Q. Then



$$| z - z_1 | = | z - z_2 |$$

$\Rightarrow$ $$| z - z_1 | = | z - z_2 |$$

$\Rightarrow$ $$(z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$$

$\Rightarrow$ $$z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z1 - z2) = |z_1|^2 - |z_2|^2.$$

Hence, R(z) lies on perpendicular bisector of $P(z_1)$ and $Q(z_2)$.

4. Equation of Straight Line

(i) Parametric form Equation of line joining points $P(z_1)$ and $Q(z_2)$ is $z = tz_1 + (1-t)z_3$, where $t \in R$

(ii) Non-Parametric form Equation of line joining points $P(z_1)$ and $Q(z_2)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

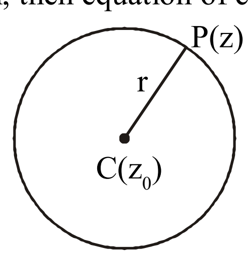
$\Rightarrow$ $$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

(iii) General equation General equation of straight line is $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.

5. Equation of a circle

(i) Equation of a circle whose radius is r and centre is $C(z_0)$, is $| z - z_0 | = r$.

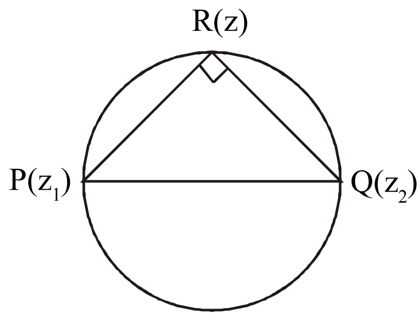
If the center of circle lies on the origin, then equation of circle is $| z | = r$.



(ii) The general equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where a e C and b e R. center of circle is at –a and radius is $\sqrt{|a|^2 - b}$.

(iii) If $P(z_1)$ and $Q(z_2)$ are the vertices of diameter of a circle, then equation of circle is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$$



6. **Some Standard Equations**

(i) The equation of parabola is

$$z\bar{z} - 4a(z + \bar{z}) = \frac{1}{2}\{z^2 + (\bar{z})^2\}.$$

(ii) The equation oa an ellipse is

$$|z - z_1| + |z - z_2| = 2a, \quad \text{Where } 2a > |z_1 - z_2|$$

and z_1 and z_2 are foci.

(iii) The equation of a hyperbola is

$$|z - z_1| - |z - z_2| = 2a, \quad \text{Where } 2a > |z_1 - z_2|$$

and z_1 and z_2 are foci.

(iv) The equation $\left|\frac{z - z_1}{z - z_2}\right| = k$ will represent a circle if $k \neq 1$.

and will represent a line if $k = 1$.

(v) The equation $|z - z_1|^2 + |z - z_2|^2 = k$ represent a circle, if $k \geq \frac{1}{2}|z_1 - z_2|^2$.

✷ **Some Important Results**

1. The coordinates of centroid of a triangle ABC whose vertex are $A(z_1)$, $B(z_2)$ and $C(z_3)$, is

$$G(z) = \frac{z_1 + z_2 + z_3}{3}$$

2. The triangle whose vertices are z1, z2, z3 is equilateral iff

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

or

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

3. If $\arg\left(\frac{(z_2-z_3)(z_1-z_4)}{(z_1-z_3)(z_2-z_4)}\right) = \pm\pi, 0$ (or purely real), then the points z_1, z_2, z_3, z_4 are concyclic.

4. $\arg\left(\frac{z-z_1}{z-z_2}\right) = 0 \Rightarrow$ Locus of z is a straight line passing through z_1 and z_2.

Note :

- $i = -\frac{1}{i}$
- The sum of any four consecutive powers of i is zero. ie,
$$i^{4n+1} + i^{4n+2} + i^{4n+3} + i^{4n+4} = 0$$
- $\sqrt{-a} = i\sqrt{a}$, when a is any real number.

  Then, $\sqrt{-a}\sqrt{-b} = i\sqrt{a}\ i\sqrt{b} = -\sqrt{ab}$

  But $\sqrt{-a}\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}$ is wrong.
- Two complex numbers cannot be compared ie, no greater complex number can be find in two given complex numbers.
- From the definition it is clear conjugate of a complex number can be obtained by replacing i by $-i$.
- If z is unimodular, then $|z| = 1$. Now, if $f(z)$ is a unimodular, then it always be expressed as $f(z) = \cos\theta + i\sin\theta, \theta \in R$.
- If $x, y \in R$, then
$$\sqrt{x+iy} + \sqrt{x-iy} = \sqrt{2\left\{\sqrt{x^2+y^2}+x\right\}}$$
$$\sqrt{x+iy} - \sqrt{x-iy} = \sqrt{2\left\{\sqrt{x^2+y^2}-x\right\}}$$
- $1 = \cos 0 + i\sin 0$
- $-1 = \cos\pi + i\sin\pi$
- $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
- $-i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
- Distance of a point P(z) from the origin $= |z|$.
- If R(z) is a mid point of PQ, then $z = \frac{z_1+z_2}{2}$

- Three points will be collinear, if for $A(z_1)$, $B(z_2)$, $C(z_3)$.

  AB + BC = AC

  ie, $|z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|$

- Three points z_1, z_2 and z_3 will be collinear, if $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

- Slop of line $\bar{a}z + a\bar{z} + b = 0$ is $-\dfrac{a}{\bar{a}}$

- If α_1 and α_2 are slopes of two lines in a complex plane, then

  (a) lines will be parallel if, $\alpha_1 = \alpha_2$.

  (b) lines will be parpendicular if, $\alpha_1 + \alpha_2 = 0$.

- Slop of line PQ joining two points $P(z_1)$ and $Q(z_2)$ = $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

- Length of perpendicular from a point $P(z_1)$ to the line $\bar{a}z + a\bar{z} + b = 0$

$$= \frac{|a\bar{z}_1 + \bar{a}z_1 + b|}{|a| + |\bar{a}|} = \frac{|a\bar{z}_1 + \bar{a}z_1 + b|}{2|\bar{a}|}$$